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DUAL INTEGRAL EQUATIONS RELATED TO THE KONTOROVICH-LEBEDEV TRANSFORM

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We study the dual integral equations related to the Kontorovich-Lebedev integral transforms arising in the course of solution of the problems of mathematical physics, in particular of the mixed boundary value problems for the wedge-shaped regions. We show that the solutions of these equations can be expressed in quadratures, using the auxilliary functions satisfying the integral Fredholm equation of second kind with a symmetric kernel.

At present, the dual equations investigated in most detail are those connected with the Fourier and Hankel integral transforms. The results obtained and their applications are given in [1 – 3]. A large number of papers also deal with the theory and applications of the dual integral equations connected with the Mehler-Fock integral transform and its generalizations [4 – 11].

The dual integral transforms considered in the present paper belong to a more complex class than those listed above, and so far, no effective solution has been obtained for them. The only relevant results known to the authors are those in [12, 13]. In [12] a method of solving the equations (1.2) is given for a single particular value of the parameter $\gamma = \pi/2$, while in [13] the dual equations of the type under consideration are reduced to a solution of an infinite system of linear algebraic equations.

1. Formulation of the problem. We consider the dual integral equations arizing in connection with the application of the Kontorovich-Lebedev transform to solving the mixed boundary value problems for wedge-shaped regions. The equations have either the form of (1.1), or of (1.2)

$$\int_0^\infty M(\tau)\,\omega(\tau)\,K_{i\tau}(\lambda r)\,d\tau = rg(r), \quad 0 < r < a \tag{1.1}$$

$$\int_0^\infty M(\tau)\,K_{i\tau}(\lambda r)\,d\tau = f(r), \quad a < r < \infty$$

$$\int_0^\infty M(\tau)\,K_{i\tau}(\lambda r)\,d\tau = f(r), \quad 0 < r < a \tag{1.2}$$

$$\int_0^\infty M(\tau)\,\omega(\tau)\,K_{i\tau}(\lambda r)\,d\tau = rg(r), \quad a < r < \infty$$

Here $(r\ \varphi\ z)$ is the system of cylindrical coordinates the z-axis of which coincides with the edge of the wedge $(0 < r < \infty,\ -\gamma < \varphi < \gamma,\ -\infty < z < \infty)$, $K_{i\tau}(\lambda r)$ is the Macdonald function with the imaginary index, $f(r)$ and $g(r)$ are given functions and

$\omega(\tau)$ is the weight function defined by the following expressions:

$$\omega(\tau) = \tau \operatorname{th} \gamma\tau, \quad \omega(\tau) = \tau \operatorname{cth} \gamma\tau \tag{1.3}$$

with one or the other expression used according to whether the boundary value problem is even or odd with respect to the variable φ. The parameter λ is assumed to be real and positive.

2. Certain discontinuous integrals containing the products of cylindrical functions. The technique of solving the dual integral equations is based on the use of discontinuous integrals the form of which is determined by the kernel of the integral transform. In the case of (1.1) and (1.2) these integrals have the form of (2.1), (2.2), (2.3) and (2.4), respectively,

$$\frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^\infty \operatorname{ch} \pi\tau \varkappa^+(\lambda t, i\tau) K_{i\tau}(\lambda r)\, d\tau = \begin{cases} \dfrac{e^{-\lambda(t-r)}}{\sqrt{\lambda(t-r)}}, & r<t \\ 0, & r>t \end{cases} \tag{2.1}$$

$$\frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^\infty \tau \operatorname{sh} \pi\tau \int_0^t \varkappa^+(\lambda s, i\tau)\, ds K_{i\tau}(\lambda r)\, d\tau = \tag{2.2}$$

$$\begin{cases} \dfrac{\sqrt{r}e^{-\lambda r}}{\sqrt{\lambda}} + \sqrt{\pi r}\Phi(\sqrt{\lambda r}), & r<t \\ \dfrac{\sqrt{r}e^{-\lambda r}}{\sqrt{\lambda}} + \sqrt{\pi r}\Phi(\sqrt{\lambda r}) - \\ -\dfrac{re^{-\lambda(r-t)}}{\sqrt{\lambda(r-t)}} - \sqrt{\pi r}\Phi(\sqrt{\lambda(r-t)}), & r>t \end{cases}$$

$$\frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^\infty \operatorname{sh} \pi\tau\, \varkappa^-(\lambda t, i\tau) K_{i\tau}(\lambda r)\, d\tau = \begin{cases} 0, & r<t \\ \dfrac{e^{-\lambda(r-t)}}{\sqrt{\lambda(r-t)}}, & r>t \end{cases} \tag{2.3}$$

$$\frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^\infty \tau \operatorname{ch} \pi\tau \int_0^t \varkappa^-(\lambda s, i\tau)\, ds K_{i\tau}(\lambda r)\, d\tau = \tag{2.4}$$

$$\begin{cases} \dfrac{re^{-\lambda(t-r)}}{\sqrt{\lambda(t-r)}} + \sqrt{\pi r}\Phi(\sqrt{\lambda(t-r)}), & r<t \\ 0, & r>t \end{cases}$$

where

$$\varkappa^+(\lambda t, i\tau) = \frac{K_{1/2+i\tau}(\lambda t) + K_{1/2-i\tau}(\lambda t)}{2}$$

$$\varkappa^-(\lambda t, i\tau) = \frac{K_{1/2+i\tau}(\lambda t) - K_{1/2-i\tau}(\lambda t)}{2i}$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2)\, ds$$

Formulas (2.1) – (2.4) appear to be novel. The formulas (2.2) – (2.4) can be proved by expanding their right-hand sides into the Kontorovich-Lebedev integral [14]

$$f(r) = \frac{2}{\pi^2} \int_0^\infty \tau \operatorname{sh} \pi\tau K_{i\tau}(\lambda r)\, d\tau \int_0^\infty \frac{f(\rho)}{\rho} K_{i\tau}(\lambda\rho)\, d\rho, \quad 0<r<\infty \tag{2.5}$$

The validity of (2.1) can be established in the analogous manner using the expansion

$$f(r)=\frac{2}{\pi^2}\int_0^\infty \tau \operatorname{sh} \pi\tau K_{i\tau}(\lambda r)\, d\tau \left[\frac{\pi f(0)}{\tau \operatorname{sh} \pi\tau}+\int_0^\infty \frac{f(\rho)-f(0)\, e^{-\lambda\rho}}{\rho} K_{i\tau}(\lambda\rho)\, d\rho\right] \qquad (2.6)$$

$$0<r<\infty$$

which generalizes the formula (2.5) to the case when $f(r)$ tends to a limit different from zero when $r \to 0$.

The expressions (2.1) – (2.4) obtained above play the same part in the theory of Eqs. (1.1) and (1.2) as the discontinuous Sonin integrals for the integral equations connected with the Hankel transform and the Mehler integrals for the equations connected with the Mehler-Fock transform.

3. Solution of the dual integral equations (1.1). In the course of solving Eqs. (1.1) we can assume the function $g(r)=0$ without loss of generality. In fact, using the substitution

$$M(\tau)=N(\tau)+P(\tau), \qquad P(\tau)=\frac{2}{\pi^2}\,\frac{\tau \operatorname{sh}\pi\tau}{\omega(\tau)}\int_0^a g(r) K_{i\tau}(\lambda r)\, dr$$

we transform the equations in question into equations of the same form in $N(\tau)$, the right-hand sides of which, by virtue of the theorem of the expansion (2.5), are

$$r\bar{g}(r)=0, \qquad \bar{f}(r)=f(r)-\int_0^\infty P(\tau) K_{i\tau}(\lambda r)\, d\tau$$

Thus, it is sufficient to investigate Eq. (1.1) for the case $g(r)=0$ and this is assumed henceforth.

We shall seek a solution of these equations in the form

$$M(\tau)=\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\,\frac{\tau \operatorname{sh}\pi\tau}{\omega(\tau)}\int_a^\infty \varphi(t)\, \varkappa^+(\lambda t, i\tau)\, dt \qquad (3.1)$$

where $\varphi(t)$ is a function continuous with its first derivative in the interval $[a, \infty)$ and tending to zero as $t\to\infty$.

Integrating by parts we obtain

$$M(\tau)=-\frac{4}{\pi\sqrt{\pi}}\,\frac{\tau \operatorname{sh}\pi\tau}{\omega(\tau)}\,\varphi(a)\int_0^a \varkappa^+(\lambda s, i\tau)\, ds - \qquad (3.2)$$

$$\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\,\frac{\tau \operatorname{sh}\pi\tau}{\omega(\tau)}\int_a^\infty \varphi'(t)\, dt\int_0^t \varkappa^+(\lambda s, i\tau)\, ds$$

Substituting (3.2) into the first equation of (1.1) (with $(g(r)=0)$ and using Eq. (2.2), we find that the equation in question is satisfied identically. The substitution of (3.1) into the second equation of (1.1) yields the integral Fredholm equation of the first kind in $\varphi(t)$

$$\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\int_a^\infty \varphi(t)\, dt\int_0^\infty \frac{\tau \operatorname{sh}\pi\tau}{\omega(\tau)}\,\varkappa^+(\lambda t, i\tau) K_{i\tau}(\lambda r)\, d\tau = f(r), \qquad a<r<\infty \quad (3.3)$$

In the case when $\omega(\tau)$ is given by (1.3), Eq. (3.3) can be transformed into an integral Fredholm equation of the second kind.

Let us assume the definiteness that $\omega(\tau)=\tau \operatorname{th}\gamma\tau$. Then

$$\frac{\tau \operatorname{sh}\pi\tau}{\omega(\tau)}=\operatorname{ch}\pi\tau+\frac{\operatorname{sh}(\pi-\gamma)\tau}{\operatorname{sh}\gamma\tau}$$

and by virtue of (2.1), Eq. (3.3) becomes

$$\int_r^\infty \varphi(t) \frac{e^{-\lambda(t-r)}}{\sqrt{\lambda(t-r)}}\, dt = F(r), \quad a < r < \infty \tag{3.4}$$

$$F(r) = f(r) - \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_a^\infty \varphi(s)\, ds \int_0^\infty \frac{\operatorname{sh}(\pi-\gamma)\tau}{\operatorname{sh}\gamma\tau} \varkappa^+(\lambda s, i\tau) K_{i\tau}(\lambda r)\, d\tau$$

Using the Abel inversion formulas we obtain

$$\varphi(t) = -\frac{\sqrt{\lambda} e^{\lambda t}}{\pi} \frac{d}{dt} \int_t^\infty F(r) \frac{e^{-\lambda r}}{\sqrt{r-t}}\, dr$$

Use of the relation

$$-\frac{e^{\lambda t}}{\sqrt{2\pi\lambda}} \frac{d}{dt} \int_t^\infty \frac{e^{-\lambda r} K_{i\tau}(\lambda r)}{\sqrt{r-t}}\, dr = \varkappa^+(\lambda t, i\tau)$$

obtained by inverting (2.2) and differentiating with respect to t, now leads to the integral Fredholm equation of the second kind with a symmetric kernel

$$\varphi(t) = -\frac{\sqrt{\lambda} e^{\lambda t}}{\pi} \frac{d}{dt} \int_t^\infty \frac{e^{-\lambda r} f(r)}{\sqrt{r-t}}\, dr - \frac{\lambda}{\pi} \int_a^\infty \varphi(s) K(s,t)\, ds, \quad a \leqslant t < \infty \tag{3.5}$$

$$K(s,t) = \frac{4}{\pi} \int_0^\infty \frac{\operatorname{sh}(\pi-\gamma)\tau}{\operatorname{sh}\gamma\tau} \varkappa^+(\lambda s, i\tau) \varkappa^+(\lambda t, i\tau)\, d\tau, \quad 0 < \gamma \leqslant \pi \tag{3.6}$$

In a similar manner we obtain the following integral equations for $\omega(\tau) = \tau \operatorname{cth} \gamma\tau$:

$$\varphi(t) = -\frac{\sqrt{\lambda} e^{\lambda t}}{\pi} \frac{d}{dt} \int_t^\infty \frac{e^{-\lambda r} f(r)}{\sqrt{r-t}}\, dr + \frac{\lambda}{\pi} \int_a^\infty \varphi(s) K(s,t)\, ds, \quad a \leqslant t < \infty \tag{3.7}$$

$$K(s,t) = \frac{4}{\pi} \int_0^\infty \frac{\operatorname{ch}(\pi-\gamma)\tau}{\operatorname{ch}\gamma\tau} \varkappa^+(\lambda s, i\tau) \varkappa^+(\lambda t, i\tau)\, d\tau, \quad 0 < \gamma < \pi \tag{3.8}$$

For certain values of the angle γ the kernels (3.6) and (3.8) can be expressed in terms of some known functions. In particular, for the kernel (3.6) this is true for all $\gamma = \pi/n$, $n = 1, 2, \ldots$. For example,

$$\begin{aligned} &\gamma = \pi, \ K(s,t) = 0 \\ &\gamma = \pi/2, \ K(s,t) = K_0(\lambda(s+t)) + K_1(\lambda(s+t)) \\ &\gamma = \pi/3, \ K(s,t) = \\ &\quad \sqrt{3} K_0(\lambda\sqrt{s^2+t^2+st}) + \frac{\sqrt{3}\,(s \mp t)}{\sqrt{s^2+t^2+st}} K_1(\lambda\sqrt{s^2+t^2+st}) \end{aligned} \tag{3.9}$$

etc.

For the kernel (3.8) analogous results are obtained for $\gamma = \pi/2n$, $n = 1, 2, \ldots$

$$\begin{aligned} &\gamma = \pi/2, \ K(s,t) = K_0(\lambda(s+t)) + K_1(\lambda(s+t)) \\ &\gamma = \pi/4, \quad K(s,t) = \sqrt{2}\, K_0(\lambda\sqrt{s^2+t^2}) + \\ &\quad \frac{\sqrt{2}(s+t)}{\sqrt{s^2+t^2}} K_1(\lambda\sqrt{s^2+t^2}) - K_0(\lambda(s+t)) - K_1(\lambda(s+t)) \end{aligned} \tag{3.10}$$

etc.

Let us now apply to the integral equations (3.5) and (3.7) an iteration process which converges rapidly when the values of the parameter λa are not too small. In particular, for $\pi / 2 \leqslant \gamma \leqslant \pi$ the formulas (3.6) and (3.8) yield the following estimates (*):

$$K^2(s, t) \leqslant [K_0(2\lambda s) + K_1(2\lambda s)]\,[K_0(2\lambda t) + K_1(2\lambda t)] \qquad (3.11)$$

$$\| K(s, t) \| \leqslant \frac{1}{\lambda} K_0\,(2\lambda a)$$

and the application of the general theory of integral equations leads to the following criterion of convergence of the iteration process:

$$K_0\,(2\,\lambda\, a) < \pi \qquad (3.12)$$

The last inequality holds when $\lambda a > 0{,}025$ and the rate of convergence of the iteration process increases with increasing λa.

When the solutions of (3.5) and (3.7) have been constructed, the solutions of the corresponding dual equations (1.1) are given by the formula (3.1).

4. Solution of the dual integral equations (1.2). In the present case we can restrict ourselves to investigating the equations in which $f(r) = 0$..The general case can be reduced to this particular case using the substitution

$$M(\tau) = N(\tau) + Q(\tau)$$

$$Q(\tau) = \frac{2}{\pi} f(0) + \frac{2}{\pi^2}\tau \operatorname{sh} \pi\tau \int\limits_0^a \frac{f(r) - f(0)\, e^{-\lambda r}}{r} K_{i\tau}(\lambda r)\, dr +$$

$$\frac{2}{\pi^2}\tau \operatorname{sh} \pi\tau \int\limits_a^\infty \frac{f(a) e^{-\lambda(r-a)} - f(0)\quad e^{-\lambda r}}{r} K_{i\tau}(\lambda r)\, dr$$

The right-hand sides of the transformed equations, on the basis of (2.6), are, respectively,

$$\bar f(r) = 0, \quad r\bar g(r) = rg(r) - \int\limits_0^\infty Q(\tau)\,\omega(\tau)\, K_{i\tau}(\lambda r)\, d\tau$$

Thus, the problem reduces to the solution of (1.2) with $f(r) = 0$. The solution of these equations is sought in the form

$$M(\tau) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \operatorname{sh} \pi\tau \int\limits_a^\infty \varphi(t)\, \varkappa^-(\lambda t, i\tau)\, dt \qquad (4.1)$$

where $\varphi(t)$ is continuous together with its first derivative on the interval $[a, \infty)$ and tends to zero for $t \to \infty$.

Using Eq. (2.3) we find, that the homogeneous equation (1.2) ($f(r) = 0$) is satisfied identically.

Let us write (4.1) in the form

$$M(\tau) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \operatorname{sh} \pi\tau \int\limits_a^\infty \varphi(t)\, d\int\limits_0^t \varkappa^-(\lambda s, i\tau)\, ds$$

*) In deriving the inequalities (3.11) we utilize the value of the integral (3.6) for $\gamma = \pi/2$ and the relation $K_0(x) \leqslant K_1(x)$.

Then, integrating by parts and substituting into the inhomogeneous equation, we obtain

$$-\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\varphi(a)\int_0^\infty \operatorname{sh}\pi\tau\,\omega(\tau)\int_0^a \varkappa^-(\lambda s, i\tau)\,ds K_{i\tau}(\lambda r)\,d\tau - \tag{4.2}$$

$$\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\int_a^\infty \varphi'(t)\,dt\int_0^\infty \operatorname{sh}\pi\tau\,\omega(\tau)\int_0^t \varkappa^-(\lambda s, i\tau)\,ds K_{i\tau}(\lambda r)\,d\tau = rg(r)$$

$$a < r < \infty$$

Subsequent calculations depend on the form of the function $\omega(\tau)$. If $\omega(\tau) = \tau \operatorname{th}\gamma\tau$, then

$$\operatorname{sh}\pi\tau\omega(\tau) = \tau\operatorname{ch}\pi\tau - \tau\frac{\operatorname{ch}(\pi-\gamma)\tau}{\operatorname{ch}\gamma\tau}$$

and Eq. (4.2) can be written, using (2.4), in the form

$$-\frac{d}{dr}e^{-\lambda r}\int_r^\infty \varphi(t)\frac{e^{-\lambda(t-r)}}{\sqrt{\lambda(t-r)}}\,dt = e^{-\lambda r}g(r) + \frac{2\sqrt{2}}{\pi\sqrt{\pi}}e^{-\lambda r}\int_a^\infty \varphi(s)\,ds \times \tag{4.3}$$

$$\int_0^\infty \tau\frac{\operatorname{ch}(\pi-\gamma)\tau}{\operatorname{ch}\gamma\tau}\varkappa^-(\lambda s, i\tau)\frac{K_{i\tau}(\lambda r)}{r}\,d\tau = e^{-\lambda r}F(r), \quad a<r<\infty$$

Integrating over the limits (r, ∞) and using the Abel inversion formulas, we obtain

$$\varphi(t) = \frac{\sqrt{\lambda}e^{\lambda t}}{\pi}\int_t^\infty \frac{e^{-\lambda r}g(r)}{\sqrt{r-t}}\,dr + \tag{4.4}$$

$$\frac{2\sqrt{2}\sqrt{\lambda}e^{\lambda t}}{\pi^2\sqrt{\pi}}\int_a^\infty \varphi(s)\,ds\int_0^\infty \tau\frac{\operatorname{ch}(\pi-\gamma)\tau}{\operatorname{ch}\gamma\tau}\varkappa^-(\lambda s, i\tau)\,d\tau\int_t^\infty \frac{e^{-\lambda r}K_{i\tau}(\lambda r)}{r\sqrt{r-t}}\,dr$$

Computing the inner integral according to the formula

$$\frac{\tau e^{\lambda t}}{\sqrt{2\pi\lambda}}\int_t^\infty \frac{e^{-\lambda r}K_{i\tau}(\lambda r)}{r\sqrt{r-t}}\,dr = \varkappa^-(\lambda t, i\tau)$$

following from (2.3), we arrive at the integral Fredholm equation of the second kind

$$\varphi(t) = \frac{\sqrt{\lambda}e^{\lambda t}}{\pi}\int_t^\infty \frac{e^{-\lambda r}g(r)}{\sqrt{r-t}}\,dr + \frac{\lambda}{\pi}\int_a^\infty \varphi(s)K(s,t)\,ds, \quad a\leqslant t<\infty \tag{4.5}$$

$$K(s,t) = \frac{4}{\pi}\int_0^\infty \frac{\operatorname{ch}(\pi-\gamma)\tau}{\operatorname{ch}\gamma\tau}\varkappa^-(\lambda s, i\tau)\varkappa^-(\lambda t, i\tau)\,d\tau, \quad 0<\gamma\leqslant\pi \tag{4.6}$$

In a similar manner we obtain, for $\omega(\tau) = \tau\operatorname{cth}\gamma\tau$

$$\varphi(t) = \frac{\sqrt{\lambda}e^{\lambda t}}{\pi}\int_t^\infty \frac{e^{-\lambda r}g(r)}{\sqrt{r-t}}\,dr - \frac{\lambda}{\pi}\int_a^\infty \varphi(s)K(s,t)\,ds, \quad a\leqslant t<\infty \tag{4.7}$$

$$K(s,t) = \frac{4}{\pi}\int_0^\infty \frac{\operatorname{sh}(\pi-\gamma)\tau}{\operatorname{sh}\gamma\tau}\varkappa^-(\lambda s, i\tau)\varkappa^-(\lambda t, i\tau)\,d\tau, \quad 0<\gamma\leqslant\pi \tag{4.8}$$

The integral equations obtained are of the same type as those in Sect. 3 and their solution can be obtained by iteration. For certain values of γ the kernels of the above equations can be expressed in terms of the known functions. Thus, e. g. the kernel of (4.5) with $\gamma = \pi/2\,n$ $(n = 1, 2, \ldots)$ and the kernel of (4.7) with $\gamma = \pi/n$ $(n = 1, 2, \ldots)$ can be expressed in a close form using the Macdonald functions.

The method of reducing the dual integral equations (1.1) and (1.2) presented in this paper can be extended to the weight functions $\omega(\tau)$ of a more general type, whose asymptotic behavior, when $\tau \to \infty$, is described by the formulas

$$\omega(\tau) \approx \tau\,\mathrm{th}\,\pi\tau, \qquad \omega(\tau) \approx \tau\,\mathrm{cth}\,\pi\tau$$

5. Application to a boundary value problem. As an example, we consider the problem of constructing a function $u = u(r, \varphi, z)$ harmonic in the region $0 < r < \infty$, $-\gamma < \varphi < \gamma$, $0 < z < l$ and satisfying the mixed boundary conditions

$$u\big|_{z=0} = u\big|_{z=l} = 0; \quad \frac{1}{r}\frac{\partial u}{\partial \varphi}\bigg|_{\substack{\varphi=\pm\gamma \\ r<a}} = 0, \quad u\big|_{\substack{\varphi=\pm\gamma \\ r>a}} = f(r, z)$$

Solution of this problem is given by the formula

$$u = \sum_{n=1}^{\infty} \sin\frac{n\pi z}{l} \int_0^{\infty} M_n(\tau)\,\frac{\mathrm{ch}\,\varphi\tau}{\mathrm{ch}\,\gamma\tau}\,K_{i\tau}\left(\frac{n\pi r}{l}\right) d\tau$$

where $M_n(\tau)$ satisfies (1.1) with $\omega(\tau) = \tau\,\mathrm{th}\,\gamma\tau$, $\lambda = n\pi/l$, $f(r) = f_n(r)$, where $f_n(r)$ are the coefficients of the Fourier expansion of the function $f(r, z)$, and $g(r) = 0$.

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ON ACCELERATION WAVES IN ANISOTROPIC THERMOELASTIC MEDIA TAKING ACCOUNT OF FINITENESS OF THE HEAT PROPAGATION VELOCITY

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The propagation of acceleration waves in an anisotropic thermoelastic medium is studied. It is shown that taking account of the finiteness of the heat distribution velocity results in the appearance of four kinds of accelaration waves, whose velocities and damping coefficients depend in an essential way on the direction of wave surface propagation. A comparison between the velocities and damping coefficients of plane acceleration waves in a zinc crystal, obtained with and without the finiteness of the heat propagation velocity taken into account, is presented.

The papers [1, 2] are devoted to the influence of the coupling of the strain and temperature fields on the nature of wave propagation in a homogeneous isotropic body in the case of an infinite heat distribution velocity. A number of features due to coupling of the fields is obtained therein, and it is shown in particular that weak and strong discontinuities damp out, and the order of damping is determined by an exponential factor.

Taking account of finiteness of the heat distribution velocity results in the appearance of two kinds of longitudinal waves whose propagation velocities depend in an essential manner on the velocity of the heat perturbation [3, 4].

1. Let us write down the system of equations governing the dynamical behavior of a thermoelastic anisotropic medium in which the heat is propagated at a finite velocity

$$q_{j,j} + c_\varepsilon\theta^\cdot + T_0\beta_{ij}\varepsilon_{ij} = 0 \tag{1.1}$$

$$\tau q_j^\cdot + q_j = -K_{ij}\theta_{,i} \tag{1.2}$$

$$\sigma_{ij,j} = \rho u_i^{\cdot\cdot} \tag{1.3}$$

$$\varepsilon_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i}) \tag{1.4}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \beta_{ij}\theta \tag{1.5}$$

Here q_j are the heat flux vector components, $\theta = T - T_0$ is the body temperature, T_0 is the body temperature in the natural state, c_ε is the specific heat for constant strain,